Elliptic Integrals, Elliptic Functions and Theta Functions
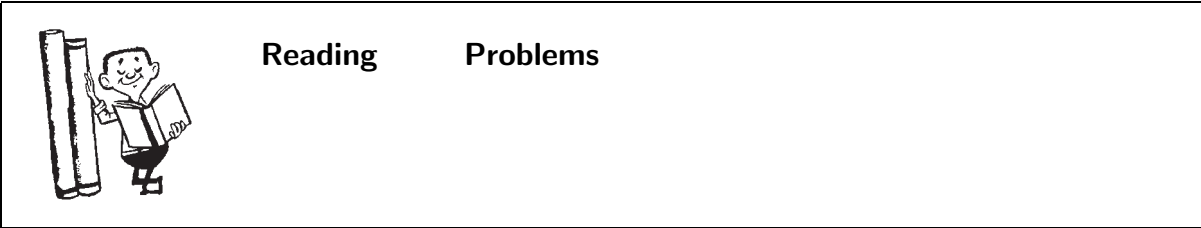
	Reading	**Problems**

Outline

Background

This chapter deals with the Legendre elliptic integrals, the Theta functions and the Jacobian elliptic functions. These elliptic integrals and functions find many applications in the theory of numbers, algebra, geometry, linear and non-linear ordinary and partial differential equations, dynamics, mechanics, electrostatics, conduction and field theory.

An elliptic integral is any integral of the general form

$$f(x) = \int \frac{A(x) + B(x)}{C(x) + D(x)\sqrt{S(x)}}\, dx$$

where $A(x), B(x), C(x)$ and $D(x)$ are polynomials in x and $S(x)$ is a polynomial of degree 3 or 4. Elliptic integrals can be viewed as generalizations of the inverse trigonometric functions. Within the scope of this course we will examine elliptic integrals of the first and second kind which take the following forms:

<u>First Kind</u>

If we let the modulus k satisfy $0 \le k^2 < 1$ (this is sometimes written in terms of the parameter $m \equiv k^2$ or modular angle $\alpha \equiv \sin^{-1} k$). The incomplete elliptic integral of the first kind is written as

$$F(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \qquad 0 \le k^2 \le 1 \quad \text{and} \quad 0 \le \sin \phi \le 1$$

if we let $t = \sin \theta$ and $dt = \cos \theta\, d\theta = \sqrt{1 - t^2}\, d\theta$, then

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \qquad 0 \le k^2 \le 1 \quad \text{and} \quad 0 \le \phi \le \pi/2$$

This is referred to as the incomplete Legendre elliptic integral. The complete elliptic integral can be obtained by setting the upper bound of the integral to its maximum range, i.e. $\sin \phi = 1$ or $\phi = \pi/2$ to give

$$\begin{aligned} K(k) &= \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \\[2mm] &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \end{aligned}$$

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<u>Second Kind</u>

$$E(\phi, k) \; = \; \int_0^{\sin \phi} \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \, dt$$

$$= \; \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

Similarly, the complete elliptic integral can be obtained by setting the upper bound of integration to the maximum value to get

$$E(k) \; = \; \int_0^1 \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \, dt$$

$$= \; \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} \, dt$$

Another very useful class of functions can be obtained by inverting the elliptic integrals. As an example of the Jacobian elliptic function sn we can write

$$u(x = \sin \phi, k) = F(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

If we wish to find the inverse of the elliptic integral

$$x = \sin \phi = sn(u, k)$$

or

$$u = \int_0^{sn} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

While there are 12 different types of Jacobian elliptic functions based on the number of poles and the upper limit on the elliptic integral, the three most popular are the copolar trio of sine amplitude, $sn(u, k)$, cosine amplitude, $cn(u, k)$ and the delta amplitude elliptic function, $dn(u, k)$ where

$$sn^2 + cn^2 = 1 \qquad \text{and} \qquad k^2 sn^2 + dn^2 = 1$$

Historical Perspective

The first reported study of elliptical integrals was in 1655 when John Wallis began to study the arc length of an ellipse. Both John Wallis (1616-1703) and Isaac Newton (1643-1727) published an infinite series expansion for the arc length of the ellipse. But it was not until the late 1700's that Legendre began to use elliptic functions for problems such as the movement of a simple pendulum and the deflection of a thin elastic bar that these types of problems could be defined in terms of *simple* functions.

Adrien-Marie Legendre (1752-1833), a French mathematician, is remembered mainly for the Legendre symbol and Legendre functions which bear his name but he spent more than forty years of his life working on elliptic functions, including the classification of elliptic integrals. His first published writings on elliptic integrals consisted of two papers in the Memoires de l'Acadmie Francaise in 1786 based on elliptic arcs. In 1792 he presented to the Acadmie a memoir on elliptic transcendents.

Legendre's major work on elliptic functions appeared in 3 volumes [5] in 1811-1816. In the first volume Legendre introduced basic properties of elliptic integrals as well as properties for beta and gamma functions. More results on beta and gamma functions appeared in the second volume together with applications of his results to mechanics, the rotation of the Earth, the attraction of ellipsoids and other problems. The third volume contained the very elaborate and now well-known tables of elliptic integrals which were calculated by Legendre himself, with an account of the mode of their construction. He then repeated much of this work again in a three volume set [6] in 1825-1830.

Despite forty years of dedication to elliptic functions, Legendre's work went essentially unnoticed by his contemporaries until 1827 when two young and as yet unknown mathematicians Abel and Jacobi placed the subject on a new basis, and revolutionized it completely.

In 1825, the Norwegian government funded Abel on a scholarly visit to France and Germany. Abel then traveled to Paris, where he gave an important paper revealing the double periodicity of the elliptic functions. Among his other accomplishments, Abel wrote a monumental work on elliptic functions[7] which unfortunately was not discovered until after his death.

Jacobi wrote the classic treatise[8] on elliptic functions, of great importance in mathematical physics, because of the need to integrate second order kinetic energy equations. The motion

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[5]Exercises du Calcul Intgral

[6]Trait des Fonctions Elliptiques

[7]Abel, N.H. "Recherches sur les fonctions elliptiques." J. reine angew. Math. 3, 160-190, 1828.

[8]Jacobi, C.G.J. Fundamentia Nova Theoriae Functionum Ellipticarum. Regiomonti, Sumtibus fratrum Borntraeger, 1829.

equations in rotational form are integrable only for the three cases of the pendulum, the symmetric top in a gravitational field, and a freely spinning body, wherein solutions are in terms of elliptic functions.

Jacobi was also the first mathematician to apply elliptic functions to number theory, for example, proving the polygonal number theorem of Pierre de Fermat. The Jacobi theta functions, frequently applied in the study of hypergeometric series, were named in his honor.

In developments of the theory of elliptic functions, modern authors mostly follow Karl Weierstrass. The notations of Weierstrass's elliptic functions based on his p-function are convenient, and any elliptic function can be expressed in terms of these. The elliptic functions introduced by Carl Jacobi, and the auxiliary theta functions (not doubly-periodic), are more complex but important both for the history and for general theory.

Theory

1. Elliptic Integrals

There are three basic forms of Legendre elliptic integrals that will be examined here; first, second and third kind. In their most general form, elliptic integrals are presented in a form referred to as incomplete integrals where the bounds of the integral representation range from
$0 \leq \sin \phi \leq 1$ or $0 \leq \phi \leq \pi/2$.

a) First Kind: The *incomplete elliptic integral* can be written as

$$\mathbf{F}(\sin \phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \quad 0 \leq k^2 \leq 1 \tag{3.1}$$

$$0 \leq \sin \phi \leq 1$$

by letting $t = \sin \theta$, Eq. 3.1 becomes

$$\mathbf{F}(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}}, \quad 0 \leq k^2 < 1 \tag{3.2}$$

$$0 \leq \phi < \frac{\pi}{2}$$

The parameter k is called the <u>modulus</u> of the elliptic integral and ϕ is the <u>amplitude</u> angle.

The *complete elliptic integral* is obtained by setting the amplitude $\phi = \pi/2$ or $\sin \phi = 1$, the maximum range on the upper bound of integration for the elliptic integral.

$$\mathbf{F}\left(\phi = \frac{\pi}{2}, k\right) = \mathbf{F}\left(\sin \phi = 1, k\right) = \mathbf{K}(k) = \mathbf{K} \tag{3.3}$$

A complementary form of the elliptical integral can be obtained by letting the modulus be

$$(k')^2 = 1 - k^2 \tag{3.4}$$

If we let $v = \tan\theta$ and in turn $dv = \sec^2\theta \, d\theta = (1 + v^2) \, d\theta$, then

$$\mathbf{F}(\phi, k) = \int_0^{\tan\phi} \frac{dv}{(1 + v^2)\sqrt{\left(1 - k^2\left(\dfrac{v^2}{1 + v^2}\right)\right)}}$$

$$= \int_0^{\tan\phi} \frac{dv}{\sqrt{1 + v^2}\sqrt{(1 + v^2 - k^2 v^2}}$$

$$= \int_0^{\tan\phi} \frac{dv}{\sqrt{(1 + v^2)(1 + k'v^2)}} \tag{3.5}$$

The complementary, complete elliptic integral can then be written as

$$\mathbf{F}\left(\phi = \frac{\pi}{2}, k'\right) = \mathbf{F}(\sin\phi = 1, k') = \mathbf{K}(k') = \mathbf{K}' \tag{3.6}$$

b) Second Kind:

$$\mathbf{E}(\phi, k) = \int_0^{\sin\phi} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt, \qquad 0 \le k^2 \le 1 \tag{3.7}$$

or its equivalent

$$\mathbf{E}(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta} \, d\theta, \qquad 0 \le k^2 \le 1 \tag{3.8}$$

$$0 \le \phi \le \frac{\pi}{2}$$

And similarly, the complete elliptic integral of the second kind can be written as

$$\mathbf{E}\left(\phi = \frac{\pi}{2}, k\right) = \mathbf{E}(\sin\phi = 1, k) = \mathbf{E}(k) = \mathbf{E} \tag{3.9}$$

and the complementary complete integral of the second kind

$$\mathbf{E}\left(\phi = \frac{\pi}{2}, k'\right) = \mathbf{E}(\sin\phi = 1, k') = \mathbf{E}(k') = \mathbf{E}' \tag{3.10}$$

c) Third Kind:

$$\Pi(\phi, n, k) = \int_0^{\sin \phi} \frac{dt}{(1 + nt)^2 \sqrt{(1 - t^2)(1 - k^2 t^2)}}, \qquad 0 \leq k^2 \leq 1$$

(3.11)

or its equivalent

$$\Pi(\phi, n, k) = \int_0^{\phi} \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{(1 - k^2 \sin^2 \theta)}}, \qquad 0 \leq k^2 \leq 1 \qquad (3.12)$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

Computer Algebra Systems

More than any other special function, you need to be very careful about the arguments you give to elliptic integrals and elliptic functions. There are several conventions in common use in competing Computer Algebra Systems and it is important that you check the documentation provided with the CAS to determine which convection is incorporated.

Function	Mathematica
Elliptic Integral of the first kind, $\mathbf{F}[\phi\|m]$ - amplitude ϕ and modulus $m = k^2$	EllipticF[ϕ,m]
Complete Elliptic Integral of the first kind, $\mathbf{K}(m)$	EllipticK[m]
Elliptic Integral of the second kind, $\mathbf{E}[\phi\|m]$ - amplitude ϕ and modulus $m = k^2$	EllipticE[ϕ,m]
Complete Elliptic Integral of the second kind, $\mathbf{E}(m)$	EllipticE[m]
Elliptic Integral of the third kind, $\mathbf{\Pi}[n;\phi\|k]$ - amplitude ϕ and modulus $m = k^2$	EllipticPi[n,ϕ,m]
Complete Elliptic Integral of the third kind, $\mathbf{\Pi}(n\|m)$	EllipticPi[n,m]

Potential Applications

Determining the arc length of a circle is easily achieved using trigonometric functions, however elliptic integrals must be used to find the arc length of an ellipse.

Tracing the arc of a pendulum can be achieved for small angles using trigonometric functions but to determine the full path of the pendulum elliptic integrals must be used.

Relations and Selected Values of Elliptic Integrals

Complete Elliptic Integrals of the First and Second Kind, K, K', E, E'

The four elliptic integrals $\mathbf{K}, \mathbf{K}', \mathbf{E},$ and $\mathbf{E}'$, satisfy the following identity attributed to Legendre

$$\mathbf{KE}' + \mathbf{K}'\mathbf{E} - \mathbf{KK}' = \frac{\pi}{2} \tag{3.13}$$

The elliptic integrals $\mathbf{K}$ and $\mathbf{E}$ as functions of the modulus k are connected by means of the following equations:

$$\frac{d\mathbf{E}}{dk} = \frac{1}{k}(\mathbf{E} - \mathbf{K}) \tag{3.14}$$

$$\frac{d\mathbf{K}}{dk} = \frac{1}{k(k')^2}[\mathbf{E} - (k')^2\mathbf{K}] \tag{3.15}$$

Incomplete Elliptic Integrals of the First and Second Kind, $F(\phi, k), E(\phi, k)$

It is convenient to introduce another frequently encountered elliptic integral which is related to $\mathbf{E}$ and $\mathbf{F}$.

$$\mathbf{D}(\phi, k) = \int_0^\phi \frac{\sin^2 \theta}{\Delta} \, d\theta = \frac{\mathbf{F} - \mathbf{E}}{k^2} \tag{3.16}$$

where

$$\Delta = \sqrt{1 - k^2 \sin^2 \theta} \tag{3.17}$$

Therefore

$$\mathbf{F} = \mathbf{E} + k^2\mathbf{D} \tag{3.18}$$

Other incomplete integrals expressed by $\mathbf{D}, \mathbf{E},$ and $\mathbf{F}$ are

$$\int_0^\phi \frac{\cos^2\theta}{\Delta}\, d\theta \;=\; \mathbf{F} - \mathbf{D} \tag{3.19}$$

$$\int_0^\phi \frac{\tan^2\theta}{\Delta}\, d\theta \;=\; \frac{\Delta\tan\phi - \mathbf{E}}{(k')^2} \tag{3.20}$$

$$\int_0^\phi \frac{d\theta}{\Delta\cos^2\theta} \;=\; \frac{\Delta\tan\phi + k^2(\mathbf{D}-\mathbf{F})}{(k')^2} \tag{3.21}$$

$$\int_0^\phi \frac{\sin^2\theta}{\Delta^2}\, d\theta \;=\; \frac{\mathbf{F}-\mathbf{D}}{(k')^2} - \frac{\sin\phi\cos\phi}{(k')^2\Delta} \tag{3.22}$$

$$\int_0^\phi \frac{\cos^2\theta}{\Delta^2}\, d\theta \;=\; \mathbf{D} + \frac{\sin\phi\cos\phi}{\Delta} \tag{3.23}$$

$$\int_0^\phi \Delta\tan^2\theta\, d\theta \;=\; \Delta\tan\phi + \mathbf{F} - 2\mathbf{E} \tag{3.24}$$

Special Values of the Elliptic Integrals

$$\mathbf{E}(0,k) \;=\; 0 \tag{3.25}$$

$$\mathbf{F}(0,k) \;=\; 0 \tag{3.26}$$

$$\pi(0,\alpha^2,k) \;=\; 0 \qquad\qquad (-\alpha^2 = n) \tag{3.27}$$

$$\mathbf{E}(\phi,k) \;=\; \phi \tag{3.28}$$

$$\mathbf{F}(\phi,k) \;=\; \phi \tag{3.29}$$

$$\pi(\phi,\alpha^2,0) \;=\; \phi \qquad\qquad (\text{if } n = 0) \tag{3.30}$$

$$\;=\; \frac{\arctan\left(\sqrt{(1-\alpha^2)}\,\tan\phi\right)}{\sqrt{1-\alpha^2}}, \qquad \text{if } \alpha^2 < 1 \tag{3.31}$$

$$\;=\; \frac{\operatorname{arctanh}\left(\sqrt{(\alpha^2-1)}\,\tan\phi\right)}{\sqrt{\alpha^2-1}}, \qquad \text{if } \alpha^2 > 1 \tag{3.32}$$

$$\mathbf{K(0)} = \mathbf{K'(1)} = \pi/2 \tag{3.33}$$

$$\mathbf{E(0)} = \mathbf{E'(1)} = \pi/2 \tag{3.34}$$

$$\mathbf{E(\phi, 1)} = \sin\phi \tag{3.35}$$

$$\mathbf{F(\phi, 1)} = \ln(\tan\phi + \sec\phi) \tag{3.36}$$

$$\pi(\phi, \alpha^2, 1) = \frac{1}{1 - \alpha^2}\left[\ln(\tan\phi + \sec\phi) - \alpha\ln\sqrt{\frac{1 + \alpha\sin\phi}{1 - \alpha\sin\phi}}\right] \tag{3.37}$$

$$\text{if } \alpha^2 > 0, \quad \alpha^2 \neq 1$$

$$= \frac{1}{1 - \alpha^2}[\ln(\tan\phi + \sec\phi) + |\alpha|\arctan(|\alpha|\sin\phi)]$$

$$\text{if } \alpha^2 < 0$$

Differentiation and Integration with Respect to the Modulus k

$$\frac{\partial \mathbf{F}}{\partial k} = \frac{k}{(k')^2}\left[\mathbf{F} - \mathbf{D} - \frac{\sin\phi\cos\phi}{\Delta}\right] \tag{3.38}$$

$$\frac{\partial \mathbf{E}}{\partial k} = -k\mathbf{D} \tag{3.39}$$

$$\frac{\partial \mathbf{D}}{\partial k} = \frac{1}{k(k')^2}\left[\mathbf{F}(\phi, k) - \mathbf{D}(\phi, k) - \frac{\sin\phi\cos\phi}{\Delta} - \frac{\mathbf{D}(\phi, k)}{k}\right] \tag{3.40}$$

$$\int \mathbf{F}\,k\,dk = \mathbf{E}(\phi, k) - (k')\mathbf{F}(\phi, k) - (1 - \Delta)\cotan\phi \tag{3.41}$$

$$\int \mathbf{D}\,k\,dk = -\mathbf{E}(\phi, k) \tag{3.42}$$

$$\int \mathbf{E}\,k\,dk = \frac{1}{3}(1 + k^2)\mathbf{E}(\phi, k) - (k')^2\mathbf{F}(\phi, k) - (1 - \Delta)\cotan\phi \tag{3.43}$$

Jacobi's Nome q and Complementary Nome q_1

The nome q and the complementary nome q_1 are defined by

$$q = q(k) = \exp[-\pi K'/K] \tag{3.44}$$

and

$$q_1 = q(k') = \exp[-\pi K/K'] \tag{3.45}$$

Therefore

$$\ln q \; \ln q_1 = \pi^2 \tag{3.46}$$

To compute the nome q we set $k = \sin\alpha$ and introduce the parameter ϵ defined by

$$2\epsilon = \frac{1 - \sqrt{\cos\alpha}}{1 + \sqrt{\cos\alpha}} \tag{3.47}$$

by means of which we have

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + 1707\epsilon^{17} \tag{3.48}$$

The above series converges rapidly provided $0 \leq \alpha \leq \pi/4$ or $0 \leq k \leq 1/\sqrt{2}$. For $\pi/4 \leq \alpha < \pi/2$ or $1/\sqrt{2} \leq k \leq 1$, let

$$2\epsilon_1 = \frac{1 - \sqrt{\sin\alpha}}{1 + \sqrt{\sin\alpha}} \tag{3.49}$$

and compute the complementary nome q_1 by means of

$$q_1 = \epsilon_1 + 2\epsilon_1^5 + 15\epsilon_1^9 + 150\epsilon_1^{13} + 1707\epsilon_1^{17} \tag{3.50}$$

2. Elliptic Functions

There are several types of elliptic functions including the Weierstrass elliptic functions as well as related theta functions but the most common elliptic functions are the Jacobian elliptic functions, based on the inverses of the three types of elliptic integrals.

1. **Jacobi elliptic functions:** The three standard forms of Jacobi elliptic integrals are denoted as $sn(u, k), cn(u, k)$ and $dn(u, k)$ and are the sine, cosine and delta amplitude elliptic functions, respectively. These functions are obtained by inverting the elliptic integral of the first kind where

$$u = F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{3.51}$$

where $0 < k^2 < 1$ and k is referred to as the elliptic modulus of u and ϕ, the upper bound on the elliptic integral is referred to as the Jacobi amplitude (amp). The inversion of the elliptic integral gives

$$\phi = F^{-1}(u, k) = amp(u, k) \tag{3.52}$$

and from this we can write

$$\sin \phi = \sin(amp(u, k)) = sn(u, k) \tag{3.53}$$

$$\cos \phi = \cos(amp(u, k)) = cn(u, k) \tag{3.54}$$

$$\sqrt{1 - k^2 \sin^2 \phi} = \sqrt{1 - k^2 \sin^2(amp(u, k))} = dn(u, k) \tag{3.55}$$

These functions are doubly periodic generalizations of the trigonometric functions satisfying

$$sn(u, 0) = \sin u \tag{3.56}$$

$$cn(u, 0) = \cos u \tag{3.57}$$
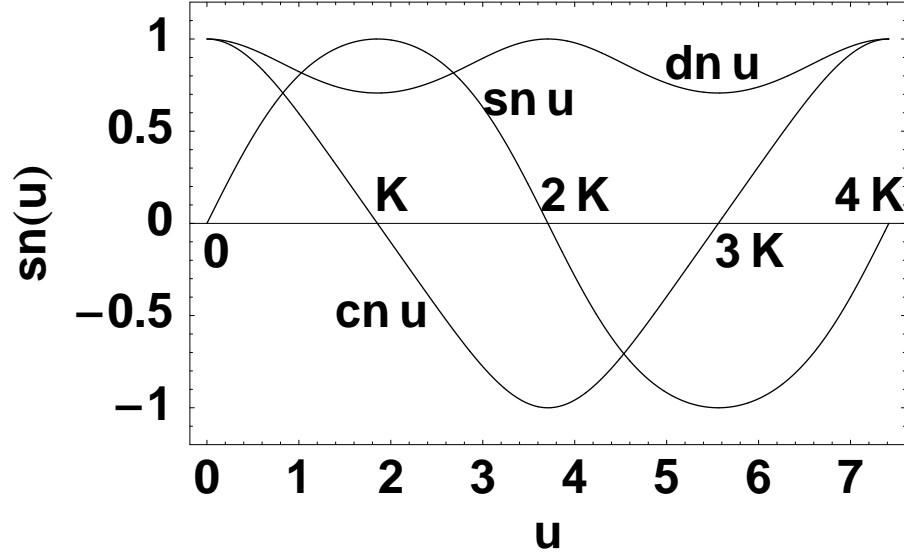
$$dn(u, 0) = 1 \tag{3.58}$$

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Figure 3.1: Plot of Jacobian Elliptic Functions $\mathbf{sn}(u)$, $\mathbf{cn}(u)$ and $\mathbf{dn}(u)$ for $k = 1/2$

In total there are 12 Jacobian elliptic functions, where the remaining 9 can be related to the 3 we have already seen as follows:

$$\mathbf{cd}(u) = \frac{\mathbf{cn}(u)}{\mathbf{dn}(u)} \qquad \mathbf{dc}(u) = \frac{\mathbf{dn}(u)}{\mathbf{cn}(u)} \qquad \mathbf{ns}(u) = \frac{1}{\mathbf{sn}(u)}$$

$$\mathbf{sd}(u) = \frac{\mathbf{sn}(u)}{\mathbf{dn}(u)} \qquad \mathbf{nc}(u) = \frac{1}{\mathbf{cn}(u)} \qquad \mathbf{ds}(u) = \frac{\mathbf{dn}(u)}{\mathbf{sn}(u)}$$

$$\mathbf{nd}(u) = \frac{1}{\mathbf{dn}(u)} \qquad \mathbf{sc}(u) = \frac{\mathbf{sn}(u)}{\mathbf{cn}(u)} \qquad \mathbf{cs}(u) = \frac{\mathbf{cn}(u)}{\mathbf{sn}(u)}$$

2. **Weierstrass elliptic functions:** The principal difference between the Jacobi and the Weierstrass elliptic integrals is in the number of poles in each fundamental cell. While the Jacobi elliptic functions has two simple poles per cell and can be considers as a solution to the differential equation

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2 + Dx^3$$

the Weierstrass elliptic function has just one double pole and is a solution to

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2$$

We will focus primarily on the Jacobi elliptic function in this course but you should be aware of the Weierstrass form of the function.

3. Theta Functions

Theta functions are the elliptic analogs of the exponential function and are typically written as, $\theta_a(u, q)$ where a ranges from 1 to 4 to represent the fours variations of the theta function, u is the argument of the function and q is the Nome, given as

$$q = e^{i\pi t} = e^{\pi K'/K} \tag{3.59}$$

where

$$t = -i\frac{K'(k)}{K(k)}$$

Assigned Problems

Problem Set for Elliptic Integrals and Functions

1. Determine the perimeter of the ellipse

$$4x^2 + 9y^2 = 36$$

2. Obtain the expression for the length of arc of the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

between $(0, 2)$ and $(1/2, \sqrt{3})$. Note that $b > a$ in this problem. Compute the arc length to three decimal places.

3. Compute to four decimal places the following integrals

   i) $\displaystyle\int_0^{\pi/4} \frac{dt}{\sqrt{1 - \dfrac{1}{2}\sin^2 t}}$

   ii) $\displaystyle\int_0^{\pi/3} \sqrt{1 - \dfrac{3}{4}\sin^2 t}\ dt$

   iii) $\displaystyle\int_0^{\pi/2} \frac{dt}{\sqrt{1 - \dfrac{1}{15}\sin^2 t}}$

   iv) $\displaystyle\int_0^{\pi/2} \sqrt{1 - \dfrac{80}{81}\sin^2 t}\ dt$

4. By means of the substitution $x = 2\sin\theta$, show that

$$\int_0^2 \frac{dx}{\sqrt{(4 - x^2)(9 - x^2)}} = \frac{1}{3}K\left(\frac{2}{3}\right)$$

5. Prove that

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \sqrt{2}\ K\left(\frac{1}{\sqrt{2}}\right)$$

6. Given $q = 1/2$, compute $k,$ $\mathbf{K}$ and $\mathbf{K}'$ to six decimal places.

7. Compute to three decimal places the area of the ellipsoid with semi-axes $\mathbf{3, 2,}$ and $\mathbf{1}$.

8. The thermal constriction resistance on an isothermal, elliptical disk $(a > b)$ on an insulated isotropic half-space of thermal conductivity k is

$$R^* = kaR = \mathbf{K}\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)$$

Compute R^* to four decimal places for the following values of $a/b : 1, 1.5, 2, 3, 4$ and 5. Use the arithmetic-geometric mean method of Gauss.

9. Derivatives of the elliptic integrals. Show that

i) $\dfrac{d\mathbf{E}}{dk} = \dfrac{\mathbf{E} - \mathbf{K}}{k}$

ii) $\dfrac{d^2\mathbf{E}}{dk^2} = -\dfrac{1}{k}\dfrac{d\mathbf{K}}{dk} = -\dfrac{\mathbf{E} - (k')^2\mathbf{K}}{k^2(k')^2}$

10. Integrals of the elliptic integrals. Show that

i) $\displaystyle\int \mathbf{K}\, dk = \dfrac{\pi k}{2}\left\{1 + \sum_{n=1}^{\infty}\dfrac{[(2n)!]^2 k^{2n}}{(2n+1)2^{4n}(n!)^4}\right\}$

ii) $\displaystyle\int k\mathbf{KL}\, dk = \mathbf{E} - (k')^2\mathbf{K}$

iii) $\displaystyle\int k\, \mathbf{E}\, dk = \dfrac{1}{3}[(1+k^2)\mathbf{E} - (k')^2\mathbf{K}]$

11. Show that

   i) $\displaystyle\int_0^1 \frac{k\mathbf{E}\,dk}{k'} = \int_0^1 \mathbf{E}(u')\,du = \frac{\pi^2}{8}$

   ii) $\displaystyle\int_0^1 \frac{\mathbf{K}\,dk}{1+k} = \frac{\pi^2}{8}$

12. Derivatives of elliptic integrals with respect to the argument

   i) $\displaystyle\frac{d}{d\phi}\,\mathbf{F}(\phi,k) = \frac{1}{\sqrt{1 - k^2\sin^2\phi}}$

   ii) $\displaystyle\frac{d}{d\phi}\,\mathbf{E}(\phi,k) = \sqrt{1 - k^2\sin^2\phi}$

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