Thermal constriction resistance with convective boundary conditions—1. Half-space contacts

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Abstract—Analytical thermal contact analysis in the past has largely been restricted to idealized boundary conditions on the contact surface. Recently, Gladwell et al. (Q. J. Mech. Appl. Math. 36(3), 387-401 (1983)) have outlined the efficient evaluation of the resulting integro-differential equation for four basic axisymmetric problems with uniform convective boundary conditions. This paper outlines in non-dimensional form, the variation of the thermal constriction resistance with Biot number for four mixed problem types on a homogeneous half-space. In addition, the thermal analysis is extended to include non-uniform flux and non-uniform convective conditions. In each case, the constriction resistance is given as a compact expression, and for several cases accurate but simpler correlations are presented.

1. INTRODUCTION

An axisymmetric boundary-value problem occurs in the classical theory of thermoelasticity when an axisymmetric heated punch, which may be circular or annular, flat or curved, contacts an elastic half-space. The uncoupled thermal problem is largely concerned with establishing the temperature field in the contact zone, satisfying the surface boundary conditions. Having determined the temperature field, one may proceed as required to evaluate stresses and displacements for the dependent elastic problem, as discussed in ref. [1].

The mathematical problem is not new, and has been studied exhaustively in the older literature [2]. For definitive treatments, see Sneddon [3], Lowengrub and Sneddon [4], Collins [5] and Sneddon [6]. However, when the surface boundary conditions are of a mixed Robin (convective) type, the resulting sets of integral equations have largely been solved numerically, as discussed by Poddubny [7], Kuz'min [8], Pavlovskii [9] and Linz [10]. These numerical techniques include the finite element [11], finite difference, and boundary element methods, and more recently, the method of moments in ref. [12]. Negus and Yovanovich [13] used the method of optimized images to examine three-dimensional conduction problems for arbitrary contacts on arbitrary flux tubes, however, with Dirichlet and Neumann boundary conditions only. Huang [14] looked at several two-dimensional, transient problems using the Weiner–Hopf technique.

Several recent investigations [15–17], have shown that application of orthogonal polynomials to mixed boundary-value problems in elasticity, yields useful and simple solutions. Gladwell et al. [18] showed hence that axisymmetric thermal problems with convective boundary conditions could be reduced to a linear, infinite set of algebraic equations. Extending this analytical integral transform procedure to include non-uniform convection coefficients and non-uniform flux distributions, this work is an extensive examination on the behaviour of the thermal constriction resistance of circular contacts on a homogeneous half-space. This thereby extends the study of thermal constriction resistance, as studied in ref. [19], to include convective boundary conditions. In the sequel Part 2, similar extensive examination is conducted for contacts on a layered half-space.

2. PROBLEM STATEMENT

Steady-state heat conduction from a circular contact to an isotropic half-space is governed by Laplace’s harmonic equation for the medium

$$\nabla^2 \Theta = 0 \quad (1)$$

where $\Theta$ is the temperature excess field ($T - T_0$). This harmonic field may be denoted, in non-dimensional circular cylinder coordinates ($\rho = r/a$, $\zeta = z/a$, $a =$ contact radius), by using the Hankel integral transform

$$\Theta(\rho, \zeta) = \mathcal{H} \int_{-\infty}^{\infty} A(\xi) \exp (-\xi \zeta) \rho d\xi \quad (2)$$

suitable only for the half-space formulation. The four types of mixed convective boundary-value problems to be studied as in ref. [18], are outlined as follows, in non-dimensional form:
NOMENCLATURE

- \( a \): contact radius dimension
- \( a_n \): Fourier series expansion coefficient and solution vector
- \( A \): standard system matrix
- \( A(\xi) \): Hankel transformed temperature function
- \( b_n, b'_n \): Fourier series expansion coefficients
- \( B \): coefficient matrix in external convection problems
- \( c \): non-uniform flux and convection distribution parameter
- \( c_1, \ldots, c_4 \): correlation parameters
- \( c_m, d_m \): Legendre series expansion coefficients
- \( d \): non-uniform flux and convection distribution parameter
- \( D \): symmetric coefficient matrix
- \( f(x) \): Fourier transform function
- \( f_{m+n} \): Fourier series coefficient
- \( g_n \): Legendre series expansion coefficient and vector
- \( h, h_1, h_2 \): convection coefficients
- \( h_{m+n} \): Fourier series coefficient
- \( H_n, H_1, H_2 \): dimensionless Biot numbers, \( h_n/k \)
- \( I \): identity matrix
- \( k \): thermal conductivity
- \( k_m, l_m \): Legendre series coefficients, \( c_m - e_m \)
- \( c_m - d_m \): integer constants
- \( n_T \): truncation value of system of equations
- \( N \): diagonal coefficient matrix
- \( P_n \): Legendre polynomial of degree \( n \)
- \( q(p), q_0(p), q_o \): heat flux functions, uniform heat flux
- \( Q, Q^* \): total heat flux through contact, total flux, \( Q/ak \)

Greek symbols

- \( \varphi, \beta_n \): Fourier series expansion coefficients
- \( \delta_{n,0} \): delta function, equals unity only for \( n = 0 \)
- \( \zeta \): dimensionless depth coordinate, \( z/a \)
- \( \Theta, \Theta_e, \Theta_o, \Theta_b \): temperature excess, mean contact temperature excess, specified base and contact temperature excesses
- \( \theta, \phi \): angular coordinate transformations
- \( \mu, \nu \): general integers
- \( \xi \): transformed radial coordinate
- \( \pi \): constant, 3.14159265...
- \( \rho \): dimensionless radial coordinate, \( r/a \)
- \( \Psi_c \): dimensionless constriction factor, \( 4akR_c \)

Other symbols

- \( \mathcal{A}, \mathcal{A}_2 \): Abel integral operator transforms
- \( \partial \): partial derivative operation
- \( \mathcal{F}, \mathcal{F}_s \): Fourier cosine and sine transform operators
- \( \mathcal{K}_v \): Hankel transform of order \( v \)
- \( \nabla \): Laplacian operator in cylindrical polar coordinates

\[
\begin{align*}
\frac{\partial \Theta}{\partial \zeta} - H_1(p)\Theta &= 0, \quad p < 1 \\
\frac{\partial \Theta}{\partial \zeta} &= 0, \quad p > 1
\end{align*}
\]

(3)

We note that \( H_1 \) and \( H_2 \) are the dimensionless Biot numbers, \( h_n/k \), corresponding to axisymmetric distributions in the internal (\( p < 1 \)) and external (\( p > 1 \)) regions, respectively. The dimensional flux \( q(p) \), defined as \( q(p)a/k \). \( \Theta_e \) and \( \Theta_b \) are respectively the specified base and contact temperature excesses. The systems to be studied are illustrated in Fig. 1.
The thermal constriction resistance of the circular contact spot on the half-space [20] is defined as

\[ R_c = \frac{\Theta_c}{Q} \tag{11} \]

where the mean contact temperature rise \( \Theta_c \), and total heat flux over the contact, \( Q \), are defined in dimensional and non-dimensional coordinates as

\[ \Theta_c = \frac{1}{\pi a^2} \int_0^\infty \Theta(r, 0) 2\pi r \, dr \tag{12} \]

\[ Q = -\int_0^\infty \frac{\partial \Theta(r, 0)}{\partial z} 2\pi r \, dr \tag{14} \]

Finally, we define the dimensionless thermal constriction resistance factor

\[ \Psi_c = 4akR_c. \tag{18} \]

Equations (3)-(10) are problem sets of dual integral equations when posed using Hankel transforms. They may be reduced to a single integro-differential equation as discussed in ref. [18], by employing appropriate Abel and Fourier transform relations. The integro-differential equation may then be reduced to an infinite linear set of algebraic equations by employing suitable Fourier expansions.

3. LIMITING SOLUTIONS

In order to validate asymptotically the solutions when \( H_1 \), or \( H_2 \), approach zero (insulated condition) or approach infinity (isothermal condition), it is important to summarize the limiting cases that are necessary.

Reference [19] has previously determined that when the external boundary is insulated, the evaluation of thermal constriction resistance for isothermal and constant flux (isoflux) contacts, is straightforward, using the Hankel representation (2). In addition, three other cases are shown in Table 1, including the two limiting problems for a symmetric contact flux distribution of the parabolic form

\[ q_0(r) = q_0(r^2 + d^2). \tag{19} \]

We note that Cases (iv) and (v) from Table 1, represent bounds on the thermal constriction resistance of a symmetric flux contact (19) with a uniform external Biot number \( H_2 \). They respectively represent the upper and lower bounds on the constriction resistance as \( H_2 \to 0 \) and \( H_2 \to \infty \).

4. PROBLEMS WITH A UNIFORM HEAT TRANSFER COEFFICIENT

By assuming a uniform convection coefficient on the contact zone or external region, the four basic problems in equations (3)-(10) may be reduced to the same infinite linear set of equations. Thus for clarity, we outline here the problem with a uniform contact conductance with external insulation, as defined in equations (3) and (4). In Hankel operator form, the dual integral equations become

\[ \mathcal{H}_d[A(\xi); \rho] + H_1 \mathcal{H}_d[\xi^{-1}A(\xi); \rho] = H_1 \Theta_0, \quad \rho < 1 \tag{20} \]

\[ \mathcal{H}_d[A(\xi); \rho] = 0, \quad \rho > 1 \tag{21} \]

where \( H_1 \) and \( \Theta_0 \) are uniform values. Condition (21) necessitates that we may represent

\[ \mathcal{F}[A(\xi); x] = (\frac{1}{\pi})^{1/2} f(x) U(1-x) \tag{22} \]

using the Abel operator \( \mathcal{A} \) [18]. \( U \) is the Heaviside unit function, defined here as unity for \( x < 1 \) and zero for \( x > 1 \). With this representation, we may reduce...
Table 1. Summary of limiting cases

<table>
<thead>
<tr>
<th>Boundary conditions on surface</th>
<th>Mean contact temperature, $\Theta_0$</th>
<th>Total heat flux, $Q^* = Q/ak$</th>
<th>Constriction factor, $4\Theta_c/Q^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho &lt; 1$</td>
<td>$\Theta_0 - \Theta_0$</td>
<td>$-\frac{q_0}{2\pi}$</td>
<td>$\frac{q_0}{2\pi}$</td>
</tr>
<tr>
<td>$\rho &gt; 1$</td>
<td>$\frac{\partial \Theta}{\partial \zeta} = 0$</td>
<td>$\frac{8q_0}{3\pi}$</td>
<td>$\frac{e}{3\pi}\frac{\pi q_0}{2}$</td>
</tr>
<tr>
<td>$\zeta = 0$</td>
<td>$\frac{\partial \Theta}{\partial \zeta} = -q_0$</td>
<td>$\frac{4q_0}{3\pi}$</td>
<td>$\frac{16}{3\pi}\frac{\pi q_0}{2}$</td>
</tr>
<tr>
<td>$\zeta = 1$</td>
<td>$\frac{\partial \Theta}{\partial \zeta} = -q_0(\rho)$</td>
<td>$\frac{8q_0}{45\pi}(7\epsilon + 15\delta)$</td>
<td>$\frac{64}{45\pi}\frac{\pi q_0}{2}(\epsilon + 2\delta)$</td>
</tr>
<tr>
<td>$\zeta = 0$</td>
<td>$\frac{\partial \Theta}{\partial \zeta} = -q_0(\rho)$</td>
<td>$\frac{4q_0}{15\pi}(2\epsilon + 5\delta)$</td>
<td>$\frac{32}{15\pi}\frac{\pi q_0}{2}(\epsilon + 2\delta)$</td>
</tr>
</tbody>
</table>

The dual equations (20) and (21) to the single Abel-type integro-differential equation

$$- \frac{1}{\rho} \frac{d}{d\rho} \int_0^\rho \frac{xf(x)dx}{(x^2 - \rho^2)^{1/2}} + H_1 \int_0^\rho \frac{f(x)dx}{(x^2 - \rho^2)^{1/2}} = -H_1 \Theta_0.$$ (23)

Integro-differential equations such as equation (23) have been exhaustively studied through the years by numerous researchers. They basically require some approximation to the function $f(x)$ and suitable selection of collocation points to ensure convergence. An excellent discussion of various procedures is given by Ioakimidis and Theocaris [21], who have summarized an excellent methodology based on Chebyshev series approximations for $f(x)$. It should be noted that equation (23) may also be represented as a Fredholm integral equation of the second kind, for which numerous solution procedures exist, as noted in earlier work [22].

We note however, from refs. [18,23], by using two simultaneous Fourier expansions for $f(x)$, that is

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n \sin (2n+1)\theta, \quad 0 < \theta < \pi/2$$ (24)

$$\sin \theta F(\theta) = \sum_{n=0}^{\infty} b_n \sin (2n+1)\theta, \quad 0 < \theta < \pi/2$$ (25)

and with the transformations $x = \cos \theta$, $\rho = \cos \phi$, then we may obtain (after employing Legendre polynomial expansions) the system

$$a_n + H_1(2n+1)^{-1}b_n = -a_n: \quad n = 0, 1, 2, \ldots$$ (26)

The $b_n$ are related to the $a_n$ through the theory of Fourier series as

$$b_n = \sum_{m=0}^{\infty} d_{mn} a_m$$ (27)

where the $d_{mn}$ are integral functions of $m$ and $n$, and as noted in ref. [18], they may be reduced in this case to closed form expressions. Furthermore, the remaining problems defined by equations (5)-(10), may also be reduced to the same infinite set (26), which satisfies regularity conditions [24], and therefore may be approximated by truncation. Further details on the derivation of the half-space integro-differential equations, may be found in refs. [18,25]. In summary, the four problems with uniform Biot numbers $H_i$, may all be reduced to the matrix system

$$\{\mathbf{a} + H_i \mathbf{N}[\mathbf{D}]\mathbf{a}\} = \mathbf{q}$$ (28)

which may be solved quite easily for the $a_n$, and the accuracy of the solution will depend on the truncation value $n_T$ used. The matrix equations for each system are summarized in the Appendix.

The necessary quantities of mean contact temperature and total heat flux through the contact (equations (12) and (14)) may be easily expressed in terms of the solution vector $a_n$ in equation (28), as found in ref. [25]. For the imperfect contact problems with uniform contact conductance, we may also express $\Theta_c$ in terms of $Q$, by multiplying the contact boundary equation by $\rho$ and integrating from $\rho = 0$ to 1. Thus, we may find

$$\Theta_c = \Theta_0 - \frac{Q^*}{H_1 \pi}$$ (29)

where

$$Q^* = \frac{\pi^2}{2} a_0$$ (30)

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$$b_n = \sum_{m=0}^{\infty} d_{mn} a_m$$ (27)

where the $d_{mn}$ are integral functions of $m$ and $n$, and as noted in ref. [18], they may be reduced in this
\[
\Psi_c = 4 \left( \frac{\Theta_0}{Q^*} - \frac{1}{H_1 \pi} \right). \tag{31}
\]

For the third basic problem, with a uniform flux \(q_0\) over the contact, and uniform external Biot number \(H_2\), we have

\[
\Theta_c = \frac{4q_0}{3\pi} + \sum_{n=1}^{\infty} a_n \left[ (-1)^{n+1} \right. \\
\times \left\{ \pi - 4 \left( \frac{1}{3} + \frac{1}{5} + \cdots + \left( \frac{1}{n-1} \right) \right) \right. \\
+ \left. \frac{4n}{4n^2 - 1} \right\} \tag{32}
\]

\[
Q^* = \pi q_0; \quad \Psi_c = 4\Theta_c/Q^*. \tag{33}
\]

Finally, for an isothermal contact with uniform external Biot number \(H_2\), we get

\[
\Theta_c = \Theta_0; \quad Q^* = 2\pi \left[ \Theta_0 + \sum_{n=1}^{\infty} a_n \left( \frac{1}{1-4n^2} \right) \right]. \tag{34}
\]

In addition, we must consider the problems defined by equations (7) and (8) when \(q_0(p)\) takes on the form of equation (19). In this case, the entries of the \(g_n\) in equation (26) are slightly modified from the form they have in ref. [18]. These are summarized in the Appendix, and upon solving the matrix equation, we may obtain

\[
Q^* = \pi q_0 \left( \frac{c}{2} + d \right); \quad \Theta_c = \frac{4q_0}{3\pi} \left( \frac{2c+d}{c} \right) + S_n \tag{35}
\]

where \(S_n\) is the series term defined previously in equation (32).

### 5. NON-UNIFORM CONDUCTANCE/CONVECTION COEFFICIENTS

By introducing a non-uniform convection or contact conductance coefficient into the basic problems outlined in the previous section, improved insights can be gained in the study of thermal constriction resistance. Since thermal contact conductance has been found in experimental work [26] to depend on the contact pressure, a variable coefficient would more accurately represent the variable pressure distribution which occurs depending upon punch geometry. Additionally, it could account for variable surface parameters such as asperity roughness, over the contact radius. Similarly, a variable external convection coefficient more accurately represents the fluid flow model outside the contact, since it is disrupted by the presence of the punch, and therefore can only be non-uniform.

First, for a non-uniform contact conductance, we choose the symmetric form

\[
h(p) = h_1(c p^2 + d) \tag{36}
\]
with \(c\) and \(d\) defined analogously as for the variable flux contact discussed earlier. In the following, we will therefore work with a redefined internal Biot number, defined as

\[
H_1 = \frac{h_1 a}{k}. \tag{37}
\]

Now, the integro-differential equation (23), expressed in terms of a series of Legendre polynomials becomes

\[
+H_1(c p^2 + d) \sum_{n=0}^{\infty} b_n P_n(2p^2-1) \hspace{1cm} \sum_{n=0}^{\infty} (2n+1)g_n P_n(2p^2-1) \tag{38}
\]

where the argument of \(P_n\) is \(w = 2p^2 - 1\).

The Legendre polynomials satisfy the well-known recurrence relation (8.914.1 in ref. [27])

\[
(n+1)P_{n+1}(w) = (2n+1)w P_n(w) - np_{n-1}(w), \quad n = 1, 2, 3, \ldots \tag{39}
\]

Using this, we may suitably approximate the non-linear component in equation (38) by

\[
w \sum_{n=0}^{\infty} b_n P_n(w) = \sum_{n=0}^{\infty} b_n P_n(w) \tag{40}
\]

where

\[
b_n = \frac{1}{2} b_1 \tag{41}
\]

\[
b_n = \frac{1}{2n-1} b_{n-1} + \frac{n+1}{2n+3} b_{n+1}; \quad n = 1, 2, 3, \ldots \tag{42}
\]

Substituting into equation (38), we thus obtain an approximate linear set

\[
a_n + H_1 \left( \frac{c}{2} + d \right) (2n+1)^{-1} b_n + H_1 \left( \frac{c}{2} + d \right) (2n+1)^{-1} b_n = g_n, \quad n = 0, 1, 2, \ldots \tag{43}
\]

The \(b_n\) are related exactly to the \(b_n\) by

\[
h_n = [G]b_n \tag{44}
\]

where \([G]\) is an \(n+1 \times (n+1)\) rectangular matrix, \(b'\) is \((n+1) \times 1\) and \(b\) is of size \((n+1) \times 1\). However, for computational purposes, we ultimately wish to represent \(b_n\) in terms of the solution vector \(a_n\), through a matrix \([D]\) defined in equation (27). If we approximate \([D]\) to be of rectangular size \((n+1) \times n\) (denoted by matrix \([D']\)), then the product will be square, i.e.

\[
h_n = [G][D']a_n \tag{45}
\]

For increasing \(n\), it was found that this rectangular approximation was stable and gave converging results, since the \(b_n\) and \(a_n\) both decrease suitably with increasing \(n\). Details on the resulting matrix equation with modified entries can be found in the Appendix. Owing to the non-uniform internal Biot number \(H_1\),
we may not use the form (29) to relate the mean contact temperature to total heat flux. Instead, we may directly express the mean contact temperature in terms of the solution vector \( a_n \). Thus from ref. [25], it may be shown that

\[
\Theta_e = \frac{(-1)^n}{2} \sum_{n=0}^{\infty} a_n \left[ \frac{1}{1-2n} + \frac{1}{3+2n} \right]
\]

(46)

where \( \mu = 0 \) for \( \Theta = 0 \) on the external boundary, and \( \mu = 1 \) for \( \partial \Theta / \partial z = 0 \) externally. For both problems, \( Q^* \) is given by equation (30).

Similarly, we can model a variable convection coefficient on the external boundary. For this we choose the form

\[
h(\rho) = h_3(\rho^{-1} + d), \quad \rho > 1.
\]

(47)

The external Biot number is now redefined by

\[
H_e = \frac{h_3 a}{k}.
\]

(48)

The necessary matrix system changes are analogous to those outlined previously for a variable contact conductance, except physically we are now studying non-uniformity on the external boundary. It is important to note that they are virtually identical for either the isoflux, isothermal and non-uniform flux-specified contact conditions.

6. PRESENTATION OF RESULTS

The standard system of equations (26) (see Appendix) was solved for all cases using a simple Gaussian elimination routine with scaled partial pivoting [28]. Convergence was governed by the truncation value \( n_T \) used for the matrix size, and this was established by specifying an accuracy on the constriction resistance of six significant figures. Generally, a larger truncation value was necessary as the reference Biot number increased, and it was found that a value of \( n_T = 50 \) gave the accuracy required for Cases (i) and (iii). A slightly larger truncation value was needed for the results in Cases (ii) and (iv), generally requiring \( n_T > 50 \) but \( n_T < 75 \). This also applied to the results for non-uniform flux and non-uniform convection conditions. The slower convergence in Cases (ii) and (iv) was expected because as the Biot number increases, the limiting solution in both cases becomes an isothermal contact with \( T = 0 \) external boundary. This is a physically unrealistic surface condition, causing a potential discontinuity at the radius \( r = a \). However, over the range of Biot numbers studied, the constriction resistance should approach smaller and smaller values, but the contact will still remain non-isothermal.

Figure 2 illustrates the variation of constriction resistance, via the dimensionless constriction factor, with the Biot number for a uniform contact conductance and insulated external boundary. The upper and lower bounds are respectively the constriction resistance for an isoflux and an isothermal contact. This result was also obtained in ref. [12]. It is interesting to note, that over the entire range of Biot numbers considered, the difference in constriction resistance amounts to only 7.5% between the upper and lower bounds. On the other hand, if the external boundary has the condition \( T = 0 \) (Fig. 2) the difference between the upper and lower bounds is 68%. It is important to point out that the results for these contacts with contact conductance provide the upper and lower bounds for when the external boundary has a uniform convective coefficient. However, this will also hold when we apply a varying external convection coefficient of the form (47). Figure 2 also shows results for when the contact is isoflux or isothermal, and an external uniform convection coefficient is applied. The upper and lower bounds for the isoflux case are respectively the upper bounds for the two previous convective contact problems. Here, the percentage difference between the upper and lower limits over the complete range of Biot numbers, is 50%. In other words, the lower bound is precisely half the upper bound. For the isothermal contact, the behaviour is slightly different, and we find that the difference is about 60% over the range of Biot numbers shown. In particular, the upper bound is specified by 1 in Table 1, which is the same as the lower bound from Fig. 2 (uniform contact conductance with external insulation).

When we apply a varying flux contact (19) and a uniform external convection coefficient, the upper and lower bounds for all cases are given in Table 1. We note that the constriction resistance is independent of the values \( c \) and \( d \) when \( c = -d \) and when \( d = 0 \). This is also the case when we had an isoflux contact, as in Fig. 2. The resulting distributions for these are similar in form to Fig. 3 except with their appropriate asymptotic limits.

However, when we superimpose flux distributions on top of a uniform flux, as in Fig. 3, we note that the constriction resistance varies for all the cases shown. From Table 1, when \( c = -d \), the difference between upper and lower bounds is about 44%, which is smaller than the isoflux case. Also, we note that the upper and lower bounds are slightly higher than the 50% for the corresponding isoflux problem. When the flux parameter \( d = 0 \), from Table 1 we see a difference between upper and lower bounds of 57%, and the bounds are lower than the isoflux problem. From Fig. 3, for the given flux variation, we see that the constriction resistance increases steadily as the superimposed flux distribution becomes larger. On the other hand, choosing \( d = 10 \), and varying \( c = 1, 10, 25, 50 \), we would find that for a larger applied heat flux, the constriction resistance steadily decreases. These are important features which may not be directly discernible from the flux distributions modelled by equation (19).

For a non-uniform contact conductance with insulated external boundary, in all cases, the lower bound
Thermal constriction resistance with convective boundary conditions

**FIG. 2.** Constriction factor vs Biot number \( H_1 \) or \( H_2 \); selected problems.

**FIG. 3.** Constriction factor vs Biot number \( H_2 \); varying flux contact with uniform external convection.

is as defined by 1 in Table 1, which is the same as observed for a uniform contact conductance shown in Fig. 2. Similarly, the upper bound in all cases corresponds to the constriction resistance of a non-uniform flux contact (with the same flux parameters (19) as the conductance parameters (36) given here) with external insulation, as noted by Table 1. Thus we begin to see the direct correspondence and symmetry that is resulting from these various solutions. Furthermore, when we have the conductance parameters defined as \( c = -d \) and \( d = 0 \), particularly useful behaviour can also be noted. If we know the constriction resistance with \( c = -1, d = 1 \), at say, a Biot number of 10, then this will be the same constriction resistance obtained for \( c = -10, d = 10 \), at a Biot number of 1. An analogous situation occurs for \( d = 0 \).

Extensive tabulations are supplied in ref. [25], from which one can predict the constriction resistance accurately for a wide range of non-uniform conductance parameters at various Biot numbers \( H_1 \). However, when a non-uniform conductance is superimposed on a given uniform conductance, the upper bound is dependent on the conductance parameters. Similar results could be shown as for a contact conductance problem with \( T = 0 \) external boundary.

Next we consider the effect of having a non-uniform external convection coefficient of the form (47). For an isoflux contact, the solutions will always remain between the upper and lower bounds defined earlier for an isoflux contact with a uniform external convection coefficient. However, depending on the convection parameters used, for intermediate Biot
numbers $H_2$ the constriction resistance will vary considerably. These differences are borne out when we compare certain cases as shown in Fig. 4, the largest difference occurring at intermediate Biot numbers $H_2$. Similar variation of constriction resistance with Biot number occurs for an isothermal contact with non-uniform external convection, as shown in Fig. 5. The upper and lower bounds again remain the same as for the uniform external convection results. If we considered a varying flux contact of the form of equation (19), and a non-uniform external convection coefficient defined by equation (47), then we could anticipate the solutions to lie within the bounds similar to Fig. 3.

Accurate and simple correlations for several cases are listed in Table 2 in a convenient hyperbolic-tangent form adopted from ref. [12]. The hyperbolic tangent is a reasonable function choice since it most closely resembles the form of the distributions generated in the figures. A non-linear least squares curve-fitting routine was used to fit the correlating form $C$ over the range $10^{-4} \leq H_i \leq 10^{4}$. The largest error associated with each fit occurs generally a decade of Biot numbers to the left or right of the inflection point seen in each distribution (i.e. Biot number of $\sim 10^3$).

As an overview, we note that for all cases studied, the Biot number was varied over the entire range $10^{-4} \leq H_i \leq 10^5$. However, not all cases approached the upper and lower bounds at the same Biot number. In Fig. 2, the distribution departs from the upper

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**Fig. 4.** Constriction factor vs Biot number $H_2$; isoflux contact with non-uniform external convection.

**Fig. 5.** Constriction factor vs Biot number $H_2$; isothermal contact with non-uniform external convection.
bound at a Biot number of $\sim 10^{-1}$ (for the contact conductance cases), whereas for external convection cases, it illustrates a departure at a Biot number of $\sim 10^{-2}$. This is particularly important in estimating the error arising from using a limiting (idealized boundary condition) solution which does not account for convection effects. If we also consider non-uniform convection on the contact zone or externally, then these departure limits are altered further. The non-uniform convective models are physically more realistic than a uniform convection coefficient, and therefore, depending on the problem type, this could significantly alter results and better explain discrepancies in experimental investigations. This is particularly important since we do not always know precisely what the boundary conditions are. Experimental experience has shown that the majority of boundary conditions are actually non-linear, and therefore the models developed here for non-uniform convection effects could provide better estimates for bounds, than the idealized conditions previously used.

### 7. CONCLUSIONS

An analytical examination has been conducted into the variation of thermal constriction resistance with variable Biot number for circular contacts on a half-space. Both uniform and non-uniform flux distributions and convective components have been included. The constriction resistance was found to vary predictably with uniform and non-uniform Biot number, between previously established asymptotic bounds in constriction resistance theory. A common trend observed was that the constriction resistance will always decrease as the reference Biot number increases, regardless of whether the convection coefficient is uniform or non-uniform, or if the convection is associated externally or on the contact zone (contact conductance). In particular, the graphical results provide a convenient check on what the actual error may be when using idealized limiting boundary conditions in a problem model. Although results were not presented for non-uniform flux contacts with non-uniform external convection conditions, the behaviour of the thermal constriction resistance can be easily predicted based on the results of this work. Additionally, the completely general problem of simultaneous convection conditions on the contact and external boundary, is the subject of an upcoming publication [28]. The bounds for this problem, may finally be established from the results shown in this study.

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APPENDIX: SOLUTION DETAILS

In the following, \([I]\) denotes the identity matrix, and \([N]\) is a diagonal matrix with entries \((2i - 1)^{-1}\) for \(i = 0, 1, \ldots, n\). Also, truncation size \(n_T = n + 1\).

Imperfect contacts (\(H\), uniform)

\[ ([I] + H_1 [N] [D]) a_i = g_i. \]  

([D] is symmetric with entries given by equations (3.17) and (4.11) in ref. [18], and for both cases of external insulation or \(T = 0\), we have

\[ g_0 = \frac{2}{n} H_1 \Theta_a, \quad g_n = 0, \quad n = 1, 2, \ldots \]  

Isothermal or flux-specified contacts (uniform \(H\))

\[ ([C] + H_2 [H]) [B]^{-1} [F] q_i = -g_i. \]  

Matrix \([C]\) is lower diagonal, with entries of \(1/2\). \([F]\) is symmetric with entries given by

\[ f_{m,n} = \frac{1}{n} \left( \frac{1}{2m+2n-1} - \frac{1}{2m+2n+1} \right) + \frac{1}{2n-2n-1} \left( \frac{1}{2m-2n+1} - \frac{1}{2m-2n-1} \right). \]

\([B]\) is upper diagonal, with explicit entries

\[ B_{ij}^{-1} = \frac{1}{B_{jj}}, \quad B_{ij}^{-1} = \frac{-B_{ij}}{B_{jj}}, \quad j = i + 1, i + 2, \ldots, n. \]

For an isoflux or isothermal contact, \(g_i/q_i\) are defined in ref. [18], and for a non-uniform flux contact of the form of equation (19), with arbitrary constants \(c\) and \(d\), from ref. [25] we have

\[ g_i = d_i q_i^0 + \frac{4c_i q_i^0}{n} \sum_{m=0}^{\infty} \frac{(-1)^m h_{m+n}}{m+n+1} \]

where the \(g_i^0\) are as defined for the isoflux case. The \(h_{m+n}\) are now given by

\[ h_{m+n} = v_{m+n} - v_{m+n-1} + \cdots + (-1)^{m+n} v_0 \]

and

\[ v_n = v_n = \frac{8}{3 \pi}. \]

The \(r_m\) are found by matching coefficients in

\[ \sum_{s=0}^{\infty} a_s w^s = \frac{\pi}{2} \sum_{s=0}^{\infty} r_m P_m(w) \]

where \(P_m\) are the Legendre polynomials, and the \(a_s\) are the power series expansion coefficients for

\[ \sin^{-1} w = \frac{(1-w^2)^{1/2}}{w^3}. \]

Non-uniform contact conductance

With the convection coefficient \(h(p)\) represented by equation (36), we have

\[ ([I] + H_1 \left( d + \frac{c}{2} \right) [N] [D] + H_1 \frac{c}{2} [N] [G] [D]) a_i = g_i. \]  

([I], [D] and [N] are as defined for the uniform conductance problems, with [D] having entries of [D] truncated to size \((n_T + 1) \times n_T\). The rectangular \((n_T \times (n_T + 1))\) matrix \([G]\) has entries from
For flux-specified contacts, the \( g_n \) are as defined for an iso-flux contact, and in the form of equation (A7) for a varying flux contact. For an isothermal contact, the resulting matrix equation becomes

\[
H_2 \frac{d}{2} \sum_{n=0}^\infty L_n P_n (2s^2 - 1) - 1 + H_c \frac{d}{2} \sum_{n=0}^\infty L_n P_n (2s^2 - 1) = \pi \sum_{n=0}^\infty (2n+1) g_n^{(1)} + g_n^{(2)} P_n (2s^2 - 1) \tag{A17}
\]

where

\[
g_n^{(1)} = 4 \pi H_2 \frac{d}{2} \sum_{n=0}^\infty \frac{(-1)^n}{m+n+1} \frac{2}{\pi} \phi_{0,0}
\]

\[
g_n^{(2)} = 2 \pi H_2 c_{1,n} (2n+1)^{-1}.
\tag{A18}
\]

or simply

\[b_n = [G] b_0. \tag{A14}\]

The \( g_n \) are

\[g_n = \frac{2}{\pi} H_1 \Theta_n \left( d + \frac{c}{2} \right), \quad g_1 = - \frac{H_1}{\pi} \Theta_1 \left( \frac{c}{3} \right)\]

and for

\[n = 2, 3, \ldots, n_T, \quad g_n = 0. \tag{A15}\]

**Non-uniform external convection**

With constants \( c \) and \( d \) defined now in equation (47), for either a flux-specified or an isothermal contact, the resulting matrix equation becomes

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots \\
1 & 0 & 2 & \cdots \\
0 & 2 & 0 & 3 \\
& 0 & 3 & 7 \\
& & \ddots & \ddots \\
& & & & \ddots \\
& & & & & 0 \\
& & & & & & \ddots \\
& & & & & & & \ddots \\
& & & & & & & & \ddots \\
& & & & & & & & & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\frac{n}{2n-1} & \frac{n+1}{2n+3}
\end{bmatrix}
\]

\[
\left\{ [C] + H_1 [N] \left( \left( d + \frac{c}{2} \right) [F] + \frac{c}{2} \frac{G}{[F]} [C] \right) \right\} a = g.
\tag{A16}
\]

**RESISTANCE THERMIQUE DE CONSTRUCTION AVEC DES CONDITIONS AUX LIMITES CONVECTIVES—1. CONTACT DE DEMI-ESPACE**

Термическое сопротивление при сжатии для конвективных граничных условий—I. Контакты в полупространстве