ANALYSIS OF THREE-DIMENSIONAL CONJUGATE HEAT TRANSFER PROBLEMS IN MICROELECTRONICS

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SUMMARY

This study is concerned with the development of an analytic-numerical method for modelling multi-layered rectangular printed circuit boards with arbitrarily located, surface mounted finite heat sources. Heat conduction is assumed to be steady-state and fully three-dimensional in the solid, with each layer having a homogeneous thermal conductivity. The cooling of the solid is achieved through a forced air flow boundary condition which includes the influence of an upstream heat flux from the solid into the fluid. An analytical Fourier series solution is used, and the boundary conditions are approximately satisfied using a continuous least-squares criterion. A linear set of symmetric equations is obtained for determining the unknown series coefficients. Unlike fully numerical, domain discretization schemes, this procedure does not require discretization, making it particularly attractive for three dimensional conduction problems. The model can simulate surface-mounted heat sources, arbitrarily located on opposite surfaces of a rectangular domain with convective cooling. Results are presented illustrating the effects of multiple laminates and heat source spacing, with applications to cooling microelectronic circuit boards.

INTRODUCTION

The thermal analysis of heat source modules arbitrarily located on the surface of rectangular multi-layered boards, is currently of practical interest to the microelectronics industry. The typical system shown in Figure 1, is comprised of a solid, multi-layered conductive board, usually of alternating reglass and copper layers, to which the electronic chip components are at-
tached. These chips can be treated as heat sources which dissipate electrical energy as heat, through the chip casing to the fluid, and via pins and surface mountings to the printed circuit board.

Of particular interest to electrical designers is the die temperature $T_D$, located within the chip module. This is unknown, but the heat dissipation rate $Q$ can be specified for the problem. The thermal resistances, between the die and fluid $R_{PF}$, and between the die and board $R_{PB}$ (see Figure 1 inset), will vary depending on package construction [1]. Experimental and numerical validations in [2] have shown that by assuming $R_{PF}$ and $R_{PB}$ to be negligible, hence $T_D = T_B = T_W$, reasonable temperature estimates can be obtained for heat source modules located in a strip-wise fashion along the board, in the direction of flow ($z$). Inherent also, is the assumption that the small module thickness does not significantly affect the flow, so that the fluid essentially sees a smooth heated wall surface.

![Diagram of 3D Thermal Board with Heat Source Modules](image)

Figure 1: 3D Thermal Board with Heat Source Modules

If the heat source modules are small relative to the overall dimensions of the solid substrate, two-dimensional analyses will not adequately predict the temperatures as noted in [3]. Also, because the board will be non-isothermal, a conjugate fluid-solid analysis must be undertaken which accounts for upstream heating of the air flow by the solid conducting medium. Theoretically this amounts to a solution of the energy-momentum equation on the fluid-
side, for its velocity \( u \) and temperature field \( T \),

\[
\frac{d}{dz} \int_{0}^{t} uTdz = -\alpha \frac{\partial T_{W}}{\partial z},
\]

which is obtained from boundary layer assumptions, and the solid-side Laplacian equation in each layer,

\[
\frac{\partial^2 T_{m}}{\partial x^2} + \frac{\partial^2 T_{m}}{\partial y^2} + \frac{\partial^2 T_{m}}{\partial z^2} = 0,
\]

with boundary conditions. Equations (1) and (2) can be simultaneously solved numerically using finite element or finite difference methods. This would involve discretization of both the fluid and solid domains requiring problem-specific mesh generation. For a smooth, non-isothermal wall, the wall-fluid temperature may be approximated \([4,5]\) using boundary layer theory approximations \([6]\) for forced convection flow, of the form

\[
T_{W} - T_{\infty} = g_{f}U_{\infty}^{-1/2}z^{-1/2} \int_{0}^{z} \frac{g_{W}(\xi) \xi^{2}}{(1 - (\xi/z)^{1/c_{1}})^{1/c_{2}}}.
\]

\( T_{\infty} \) and \( U_{\infty} \) are free stream quantities, and \( g_{f}, c_{1} \), and \( c_{2} \) vary depending on whether the flow is laminar or turbulent, wall shear conditions, and assumed velocity profiles \([4,5]\). Numerous researchers have studied similar problems with different assumptions and methods \([7-11]\). A film coefficient can usually be extracted from (3) for use in the classical convection boundary condition needed for the exposed surfaces;

\[
-k \frac{\partial T_{W}}{\partial z} - h(x, T_{W})(T_{W} - T_{\infty}) + q(x, y) = 0.
\]

The film coefficient \( h \) and heat flux \( q \) in (4) may both vary positionally, but since \( h \) will also be temperature dependent, it must be iteratively computed using (3) and a solution to (2). Models using (4) automatically require iteration to establish \( h \), and are also automatically restricted to discretizing the entire exposed surface. Although the method to be introduced here can easily incorporate the form (4) as shown in \([12,13]\), this study directly uses the interface condition (3) thereby avoiding unnecessary intermediate iterations and discretization.

. PRELIMINARY ANALYSIS

The origin of the local coordinate systems, in \( x, y, z \) of each layer, located at the bottom left corner as illustrated in Figure 1 for layer 1. Setting \( \theta_{m} \equiv T_{m}(x, y, z) - T_{\infty} \), with homogeneous thermal conductivity \( k_{m} \), separable solution can be obtained in a straightforward manner,

\[
\theta_{m}(x, y, z) = c_{m}x + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \cos(\lambda_{i}x) \cos(\beta_{j}y)[a_{m,i,j} \cosh(\gamma_{i,j}z) +

+b_{m,i,j} \sinh(\gamma_{i,j}z)],
\]
which satisfies the insulated end conditions for each layer at \( z = 0, L_1 \), and \( y = 0, L_2 \), with constants

\[
\lambda_i = \frac{i\pi}{L_1}, \quad \beta_j = \frac{j\pi}{L_2}, \quad \gamma_{i,j} = \sqrt{\lambda_i^2 + \beta_j^2}.
\]

The board ends are treated as insulated since the aspect ratio of the exposed \( z-y \) surface to the total \( z \)-thickness is generally \( > 100 : 1 \). Each of the layers are in perfect contact with each other, thus satisfying the following boundary conditions along the whole of each interface,

\[
\theta_m(x, y, t_m) = \theta_{m+1}(x, y, 0), \quad \kappa_m \frac{\partial \theta_m}{\partial x}(x, y, t_m) = \frac{\partial \theta_{m+1}}{\partial x}(x, y, 0); \quad \kappa_m = \frac{k_m}{k_{m+1}}.
\]

Using orthogonality, the constants in (5) may be related using (7) and (8), and it can be shown that in proceeding from layer 1 to layer \( M \),

\[
a_{m+1,i,j} = a_{m,i,j} \cosh(\gamma_{i,j} t_m) + b_{m,i,j} \sinh(\gamma_{i,j} t_m) \quad ; i,j = 0,1,2,\ldots
\]

\[
b_{m+1,i,j} = \kappa_m (a_{m,i,j} \sinh(\gamma_{i,j} t_m) + b_{m,i,j} \cosh(\gamma_{i,j} t_m)) \quad ; i,j = 0,1,2,\ldots
\]

\[
a_{m+1,0,0} = a_{m,0,0} + c_{m} t_m; \quad c_{m+1} = \kappa_m c_m.
\]

In (9) and (10), the relations hold whenever \( i \) and \( j \) are both not zero. Tables and relations are provided in [14] for three types of unmixed boundary conditions imposed on layer 1 at \( z = 0 \) (surface \( S_1 \)), which initialize the first layer constants for use in (9)-(11). The special case where \( S_1 \) also has mixed boundary conditions with multiple heat flux and convective sections, is considered later in section 4.

Using (5)-(11), we can write the general series form on surface \( S_2 \) (layer \( M \), \( z = t_M \)),

\[
\theta_M(x, y, t_M) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (d_{i,j} \phi_{i,j}(t_M) + \sigma_{i,j}(t_M)) \cos(\lambda_i x) \cos(\beta_j y),
\]

\[
\frac{\partial \theta_M}{\partial x}(x, y, t_M) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (d_{i,j} \phi'_{i,j}(t_M) + \sigma'_{i,j}(t_M)) \cos(\lambda_i x) \cos(\beta_j y),
\]

with

\[
\phi_{i,j}(x) = F_{1,i,j} \cosh(\gamma_{i,j} x) + F_{2,i,j} \sinh(\gamma_{i,j} x) \quad ; i,j = 0,1,2,\ldots
\]

\[
\sigma_{i,j}(x) = F_{5,i,j} \cosh(\gamma_{i,j} x) + F_{6,i,j} \sinh(\gamma_{i,j} x) \quad ; i,j = 0,1,2,\ldots
\]

\[
\phi_{0,0}(x) = F_3 x + F_4; \quad \sigma_{0,0}(x) = F_7 x + F_8
\]

The derivative functions \( \phi' \) and \( \sigma' \) are obtained by differentiating the above with respect to \( x \). The \( F_i \) along with the \( d_{i,j} \) are outlined in [14]. For
clarity, the \( d_{i,j} \) are expressible in terms of \( a_{i,j}, b_{i,j} \) and \( c \), depending on the \( S_1 \) conditions.

From the fluid-side, the general wall condition to be satisfied is given by (3). We shall consider \( R_{PP} = R_{PB} = 0 \); thus the form (3) is continuous along the wall-fluid interface, i.e. on \( S_2 \), \( T_W(x) - T_\infty \equiv \theta_M(x, y, t_M) \). The form (3) also allows for surfaces \( S_1 \) and \( S_2 \) to have independent fluid free-stream properties through the constant \( g_f \).

3. SOLUTIONS FOR MIXED CONDITIONS ON \( S_2 \) ONLY

Since \( R_{PP} = R_{PB} = 0 \), the positive heat flux entering the fluid from surface \( S_2 \), for use in (3), may be written as

\[
q_{w,f}(x) = -k_M \frac{\partial \theta_M(x, y, t_M)}{\partial z} + q_{s2} \, ,
\]

where \( q_{s2} \) may take on finite or zero values across the exposed surface. Substituting (12), (13) and (17) into (3), we obtain after a little manipulation,

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{i,j} \psi_{i,j}(z, y) - f(x, y) = 0 \, ,
\]

where

\[
d_{i,j} = d_{i,j} F_{i,j} \cosh(\gamma_i t_M) \, , \quad i, j = 0, 1, 2, \ldots \quad \text{and} \quad d_{0,0} = d_{0,0} \]

\[
\psi_{i,j}(x, y) = \cos(\beta_j y) \left( \rho_{i,j}(t_M) \cos(\lambda_i x) + k_M \rho'_{i,j}(t_M) \int_0^\infty g_2(\xi, x) \cos(\lambda_i \xi) d\xi \right) \]

\[
\rho_{i,j}(t_M) = 1 + F_{2,i,j} \tanh(\gamma_i t_M) \, ; \, \rho_{0,0}(t_M) = \rho_{0,0}(t_M) \]

\[
\rho'_{i,j}(t_M) = \gamma_i \left( F_{2,i,j} + \tanh(\gamma_i t_M) \right) \, ; \, \rho'_{0,0}(t_M) = \rho'_{0,0}(t_M) \]

\[
f(x, y) = \int_0^2 g_2(\xi, x) q_{s2} d\xi - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \cos(\beta_j y) \left\{ a_{i,j}(t_M) \cos(\lambda_i x) \right\} + c'_{i,j}(t_M) k_M \int_0^2 g_2(\xi, x) \cos(\lambda_i \xi) d\xi \, .
\]

We define the quadratic functional on \( S_2 \),

\[
I_{S2} = \int_0^{L_1} \int_0^{L_2} \left[ \sum_{i=0}^{N} \sum_{j=0}^{N} d_{i,j} \psi_{i,j}(x, y) - f(x, y) \right]^2 dx dy \, .
\]

By minimizing this variational form, a system of linear equations results which may be used for the solution of a finite number of unknown \( d_{i,j} \).

Thus, taking \( \partial I_{S2} / \partial d'_{p,n} = 0 \), we obtain the \((N + 1)^2\) linear equations

\[
Rd' = G \, ,
\]
with

\[ R_{p,n,i,j} = \sum_{s=1}^{j_s} \int_{s} \psi_{p,n}(x,y) \psi_{i,j}(x,y) \, dx \, dy \; ; \; p,n,i,j = 0,1,2,\ldots N, \quad (26) \]

\[ G_{p,n} = \sum_{s=1}^{j_s} \int_{S} \psi_{p,n}(x,y) f(x,y) \, dx \, dy \; ; \; p,n = 0,1,2,\ldots N. \quad (27) \]

where \( \int \) denotes an area integration over every section \( s \) on \( S_2 \), where \( \psi \) or \( f \) may change value with \( x \) and \( y \). Since (3) is positionally dependent on coordinate \( z \), and not \( y \), the total number of sections \( J_s \) necessary is only dependent on the number of heat source locations [14].

Solving (25) will yield an approximate set of \((N+1)^2\) coefficients \( d_{i,j}' \), which can be post-processed to yield the necessary solution coefficients in (5) for the thermal fields. However because of the earlier assumptions that \( T_W = T_B = T_D \), the integrations in (26) can be singly represented by a total area integration \( \int_{0}^{L_1} \int_{0}^{L_2} \) over \( S_2 \), regardless of heat source location, since \( \psi \) is unaffected. The \( R \) matrix can be computed once for a given problem. Further details on particular integrations can be found in [14].

4. MIXED BOUNDARY CONDITIONS ON \( S_1 AND S_2 \)

Here we assume that heat flux specified sections with forced convection cooling, are also arbitrarily located on both surfaces \( S_1 \) and \( S_2 \). This is representative of electronic circuit boards with chip modules mounted on sides of the board. This results in two unknown coefficients which have to be determined in (5), as opposed to one in (12). A procedure detailed in [12,13] can be used to relate coefficients in (5) in a similar manner that was shown for the solution of the coefficients in section 3. The location and intensity of heat sources, including fluid properties, may differ on \( S_2 \) from \( S_1 \). We will briefly outline the methodology in the following.

The general condition on surface \( S_1 \), with forced convection and arbitrarily placed heat sources, may be written as

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} \psi_{i,j}(x,y) + b_{i,j} \psi_{i,j}(x,y) - f_1(x,y) = 0, \quad (28) \]

where \( \psi_1 \) and \( \psi_2 \) can be established using a similar methodology as in section 3 to derive (18), and outlined in detail in [14]. We can then define on \( S_1 \) the form

\[ I_{S_1} = \int_{0}^{L_1} \int_{0}^{L_2} \left[ \sum_{i=0}^{N} \sum_{j=0}^{N} a_{i,j} \psi_{i,j}(x,y) + b_{i,j} \psi_{i,j}(x,y) - f_1(x,y) \right]^2 \, dx \, dy, \quad (29) \]
and on $S_2$

$$I_{S_2} = \int_0^{L_1} \int_0^{L_2} \left[ \sum_{i=0}^{N} \sum_{j=0}^{N} a_{M,i,j} \psi_{3,i,j}(x,y) + b_{M,i,j} \psi_4(3,i,j)(x,y) - f_M(x,y) \right]^2 dxdy,$$

(30)

or

$$I_{S_2} = \int_0^{L_1} \int_0^{L_2} \left[ \sum_{i=0}^{N} \sum_{j=0}^{N} a_{1,i,j} \psi_{5,i,j}(x,y) + b_{1,i,j} \psi_6(1,i,j)(x,y) - f_2(x,y) \right]\ dxdy,$$

(31)

where $\psi_3, \psi_4$ can be related recursively to $\psi_2$ and $\psi_4$ using a simple algorithm based on the forms (9)-(11). We now take two minimizations of forms (29) and (31), i.e. let $I = I_{S_1} + I_{S_2}$, thus we take

$$\frac{\partial I}{\partial a_{1,p,n}} = 0, \quad \frac{\partial I}{\partial b_{1,p,n}} = 0,$$

(32)

to obtain the $2(N+1)^2$ equations

$$\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},$$

(33)

for the solution of $a_{1,i,j}$ and $b_{1,i,j}$. Particular entries of (33) are similar in form to those in (26), and are detailed in [14].

5. NUMERICAL RESULTS AND DISCUSSION

Numerical cases were studied examining the effect of heat source location on a laminated, 7-layer board. Figure 2 illustrates results for a single heat source centrally located (see inset) on the top board surface $S_2$, with forced convection cooling (3) on both sides, $S_1$ and $S_2$, of the board. Parameters are given in Table 1. The dimensionless heat source $x$-width $/L_1$, $W_x$, was fixed at 0.2, and the $y$-width $/L_2$, $W_y$, was varied between a small hot-spot size of 0.2 to a strip size of 1.0.

Similarly, Figures 3 and 4 show results for two heat sources of different size; a larger heat source trailing the first, and the larger preceding the second, in the $x$-direction of coolant flow. All figures show the temperatures of the top surface $S_2$, from $x = 0$ to $L_1$, along a constant $y$-line located centrally on the source (denoted by - - - - in the insets). In all cases a two-dimensional $x-z$ strip solution would correspond to $W_y = 1$; a two-dimensional $x-y$ spatial solution, denoted by — — in the graphs, corresponds to a line of symmetry (insulated boundary) being drawn midway through the total $z$-thickness of the board (and $q \equiv q/2$). This $x-y$ spatial solution is also sometimes determined by assuming an effective thermal conductivity of the substrate, thereby treating the $z$-temperature drop as negligible.
Differences can be noted between the possible two-dimensional approximations to the three-dimensional thermal spreading problem, and these are always largest at the heat source. In all instances, the spatial solution will underestimate, and the strip solution will overestimate, the temperatures in the system. Further results are presented in [14].

The continuous least-squares Fourier solution to the heat conduction problem has been shown ([12,13]) to give stable estimates for the coefficients as the truncation value \( N \) is increased. As is typical with Fourier series type solutions, this value controls the level of error in convergence, as opposed to the degree of discretization required with numerical methods such as finite element, finite difference and even the boundary element method. For all cases considered herein, it was found that at most, a 5% temperature difference was observed between taking a truncation value as little as \( N = 10 \), compared with \( N > 50 \), to which the temperature solution could be converged to higher decimal degree of accuracy, but with significantly increased computation time.
Board

(Alumilase) \hspace{2cm} (copper)

$t_1, t_3, t_5, t_7 = 3.8 \times 10^{-4} \text{ m}$ \hspace{1cm} $t_4 = 3.6 \times 10^{-5} \text{ m}$

$k_1, k_3, k_5, k_7 = 0.4 \text{ W/mK}$ \hspace{1cm} $t_2, t_6, t_8 = 1.3 \times 10^{-5} \text{ m}$

$L_1 = L_2 = 0.1 \text{ m}$ \hspace{1cm} $k_2, k_4, k_6 = 386 \text{ W/mK}$

Fluid (Air-laminar flow) \hspace{1cm} $g_f = 0.73P_{r}^{-1/3\nu^{1/2}k_f^{-1}} m^2 K/s^{1/2} W$

$\bar{U}_\infty = 3 \text{ m/s} \hspace{1cm} T_\infty = 20^\circ C$

$k_{f,1} = k_{f,2} = 2.6 \times 10^{-2} \text{ W/mK} \hspace{1cm} Pr = 0.71$

$c_1 = 1 \hspace{1cm} c_2 = 3/2 \hspace{1cm} \nu = 15 \times 10^{-6} \text{ m}^2/s$

Heat Source: centroids at $x^* = x/L_1 \hspace{1cm} y^* = y/L_2$

**Figure 2** \hspace{2cm} $x^* = y^* = 0.5 \hspace{1cm} W_{z1} = 0.2 \hspace{1cm} W_{y} = 0.2, 0.4, 0.7, 1.0$

$q = 5000 \text{ W/m}^2$

**Figure 3** \hspace{2cm} $W_{z1} = 0.1 \hspace{1cm} W_{y1} = 0.1, 1.0; \hspace{1cm} x^* = 0.15; \hspace{1cm} y^* = 0.85, 0.5$

$W_{z2} = 0.4 \hspace{1cm} W_{y2} = 0.4, 1.0; \hspace{1cm} x^* = 0.60; \hspace{1cm} y^* = 0.40, 0.5$

$q_1 = 1000 \text{ W/m}^2 \hspace{1cm} q_2 = 4000 \text{ W/m}^2$

**Figure 4** \hspace{2cm} $W_{z1} = 0.4 \hspace{1cm} W_{y1} = 0.4, 1.0; \hspace{1cm} x^* = 0.40; \hspace{1cm} y^* = 0.60, 0.5$

$W_{z2} = 0.1 \hspace{1cm} W_{y2} = 0.1, 1.0; \hspace{1cm} x^* = 0.85; \hspace{1cm} y^* = 0.15, 0.5$

$q_1 = 4000 \text{ W/m}^2 \hspace{1cm} q_2 = 1000 \text{ W/m}^2$

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Table 1: Parameters for Numerical Studies

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{z1}$</td>
<td>S1 Plan View</td>
</tr>
<tr>
<td>$w_{y}$</td>
<td>3D</td>
</tr>
<tr>
<td>$q_1$</td>
<td>2D (spatial)</td>
</tr>
</tbody>
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**Figure 3**: Effects of a Larger Downstream Heat Source
Figure 4: Effects of a Larger Upstream Heat Source

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